

Contributions to the compositional semantics of first-order predicate logic

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Abstract

Henkin, Monk and Tarski gave a compositional semantics for first-order predicate logic. We extend this work by including function symbols in the language and by giving the denotation of the atomic formula as a composition of the denotations of its predicate symbol and of its tuple of arguments. In addition we give the denotation of a term as a composition of the denotations of its function symbol and of its tuple of arguments.

1 Introduction

To introduce compositional semantics we can do no better than to quote [6]:

The standard interpretation of compositionality is that for basic expressions a meaning is given, and that operations are defined on these meanings which yield meanings for compound expressions. Almost all modern linguistic theories which give serious attention to semantics follow this idea, [...]

First-order predicate logic is one of the linguistic theories where serious attention is given to semantics. As the language of this logic is formal, its semantics should, *a fortiori*, be compositional. The first semantics, [12] ([14] in German, [13] in English) was not compositional. The first compositional semantics ([15], [3]) omits function symbols from the language and stops short of providing a compositional semantics of atomic formulas and of terms. In [17] this work was extended by including function symbols in the language. It was also extended by giving the denotation of the atomic formula as a composition of the denotations of its predicate symbol and of its tuple of arguments. However, [17] was restricted to logic programs with Herbrand semantics. In the present paper we extend this work in several directions.

- We remove the restriction to Herbrand semantics and make the results applicable to arbitrary interpretations.
- We remove the restriction to logic programs
- We not only provide a compositional semantics for atomic formulas, but also for composite terms.

The initial impetus for the present paper was to give compositional semantics for terms and for atomic formulas by relating the meaning of the function or predicate symbols to the meanings of the tuples of the terms that occur as arguments. In pursuit of this purpose, interpretations acquire a central position. Once we obtained our basic results we needed

to link these to the compositional semantics already established in [3]. The presence of interpretations and function symbols forced us to deviate from their method. The result is a compositional semantics for first-order predicate logic that integrates our new results with a reformulation of the pre-existing ones. Although this reformulation is a minor departure from [3], we believe it has some interest in its own right.

2 Sets, functions, tuples, and relations

In this section we establish the notation and terminology for basic notions concerning sets, functions, tuples, and relations.

2.1 Sets

We assume the sets of standard (Zermelo-Fraenkel) set theory. The sets specified in this paper owe their existence to the axiom of specification: S' defined as equal to $\{x \in S \mid F\}$ where F is an expression of which the truth depends of x . In spite of the axiom of extensionality, according to which a set is entirely determined by its elements, we distinguish S' from $\{x \in T \mid F\}$ even when $T \supseteq S$.

We denote the cardinality of a set V by $|V|$. For a finite V with $|V| = n$, we freely confuse the finite cardinals with the corresponding ordinals and loosely refer to them as “natural” numbers. As a result, locutions such as “for all $i \in n$ ” are common as abbreviation of “for all $i \in \{0, \dots, n-1\}$ ”.

2.2 Functions

The set of all functions from set S to set T , not both empty¹, is denoted $S \rightarrow T$, so that we may write $f \in (S \rightarrow T)$. To relieve the overloaded term “domain” we call S the *source* and T the *target* of $S \rightarrow T$ and of any f belonging to it. All functions have one argument. But the source may be a set that is composed of other sets.

The value of f at $x \in S$ is written as fx or as f_x . The process of ascertaining the value of f at argument x is called the *application* of f to x . We often write “ $f(x)$ ” instead of “ fx ”; this can help clarify what gets applied to what.

The composition h of $f \in (S \rightarrow T)$ and $g \in (T \rightarrow U)$ is written as $g \circ f$ and is the function in $S \rightarrow U$ defined by $x \mapsto g(f(x))$ for all $x \in S$. It is often convenient to interchange the order of the arguments in $g \circ f$. In such situations we write $g \circ f$ as $f \triangleright g$.

Definition 1 Given $f \in (S \rightarrow T)$ we also use f to denote the set extension of f , which is the function in $2^S \rightarrow 2^T$ with map $s \mapsto \{f(x) \mid x \in s\}$.

The inverse set extension f^{-1} of f is the function in $2^T \rightarrow 2^S$ with map $t \mapsto \{x \in S \mid f(x) \in t\}$.

Lemma 1 Let f be a function in $S \rightarrow T$. We have² $s \subseteq f^{-1}(f(s))$ for all $s \subseteq S$. We have $t = f(f^{-1}(t))$ for all $t \subseteq T$.

Definition 2 Let $f \in (S \rightarrow T)$ and let S' be a subset of S . The restriction $f \downarrow S'$ of f to S' is the function in $S' \rightarrow T$ with map $x \mapsto f(x)$. \square

¹ Our reason for this restriction is that we want $|S \rightarrow T| = |T|^{|S|}$ to hold, and that this is defined except when both S and T are empty. Thus S may be empty. In that case $S \rightarrow T$ contains one function, which we could call the empty function with target T . Formally there is a different such function for every different T .

² That we do not have equality between the sets is necessitated by the possibility that f is not injective.

2.3 Tuples

Tuples are regarded as functions. When a function is a tuple, then the source of the function is often referred to as its “index set”. The tuple $t \in (n \rightarrow U)$, with the index set n a natural number, can be written as (t_0, \dots, t_{n-1}) . Tuples can also be indexed by other sets. For example, consider a certain tuple $f \in (\{x, y, z\} \rightarrow \{0, 1\})$ specified by $f(x) = 0$, $f(y) = 1$, and $f(z) = 0$. We can achieve some abbreviation by listing the graph of f : writing e.g. $f = \{\langle z, 0 \rangle, \langle x, 0 \rangle, \langle y, 1 \rangle\}$. Another possibility is a tabular representation of this tuple: $f = \frac{x \mid y \mid z}{0 \mid 1 \mid 0}$. This has the advantage of a compact representation of a set

of tuples of the same type: $\frac{x \mid y \mid z}{0 \mid 1 \mid 0} \quad \frac{x \mid y \mid z}{1 \mid 0 \mid 1}$.

The source S of $f \in (S \rightarrow T)$ may be a set of tuples, for example the set $n \rightarrow U$. The value of f at $t \in (n \rightarrow U)$ can be written $f(t_0, \dots, t_{n-1})$ as alternative to ft .

2.4 Relations

Definition 3 1. A relation R is a triple $\langle I, D, C \rangle$ of sets satisfying the constraint that $C \subseteq (I \rightarrow D)$. $I \rightarrow D$ is the domain of the relation, I is the index set of the relation, and C is the content of the relation.

2. When two relations have the same index set and the same domain, some of the common operations on and relations between their contents can be extended:

$$\begin{aligned} \langle I, D, C_0 \rangle \subseteq \langle I, D, C_1 \rangle & \text{ iff } C_0 \subseteq C_1 \\ \langle I, D, C_0 \rangle \cap \langle I, D, C_1 \rangle & = \langle I, D, C_0 \cap C_1 \rangle \\ \langle I, D, C_0 \rangle \cup \langle I, D, C_1 \rangle & = \langle I, D, C_0 \cup C_1 \rangle \\ \langle I, D, C_0 \rangle \setminus \langle I, D, C_1 \rangle & = \langle I, D, C_0 \setminus C_1 \rangle \\ \langle I, D, C \rangle^\sim & = \langle I, D, (I \rightarrow D) \setminus C \rangle \end{aligned}$$

3. Let $R_0 = \langle I_0, D, C_0 \rangle$ and $R_1 = \langle I_1, D, C_1 \rangle$. We write the bowtie of R_0 and R_1 as $R_0 \bowtie R_1$ and define it to be $\langle I_0 \cup I_1, D, C \rangle$ where

$$C = \{r \in ((I_0 \cup I_1) \rightarrow D) \mid (r \downarrow I_0) \in R_0 \text{ and } (r \downarrow I_1) \in R_1\}.$$

Similarity for the oplus of R_0 and R_1 , which is defined to be $\langle I_0 \cup I_1, D, C \rangle$ where

$$C = \{r \in ((I_0 \cup I_1) \rightarrow D) \mid (r \downarrow I_0) \in R_0 \text{ or } (r \downarrow I_1) \in R_1\}.$$

4. Let $R = \langle I, D, C \rangle$ be a relation and let I' be a subset of I . Let $R' = \langle I', D, C' \rangle$ be a relation. We write the projection of R on I' as $\pi_{I'}(R)$ and define it to be

$$\langle I', D, \{c \downarrow I' \mid c \in C\} \rangle.$$

We write the cylinder in I on R' as $\rho_I(R')$ and define it to be

$$\langle I, D, \{c \in (I \rightarrow D) \mid (c \downarrow I') \in C'\} \rangle.$$

□

When referring to a relation, its index set and domain are often clear from the context. In such cases we refer to the relation by its content only.

Example 1 Let $I_0 = \{x, y\}$, $I_1 = \{y, z\}$, and $D = \{a, b\}$, let $R_0 = \langle I_0, D, \frac{x}{a} \mid \frac{y}{b} \rangle$ and

let $R_1 = \langle I_1, D, \frac{y}{a} \mid \frac{z}{b} \rangle$.

We have $R_0 \bowtie R_1 = \langle \{x, y, z\}, D, \frac{x}{a} \mid \frac{y}{b} \mid \frac{z}{a} \rangle$. We have $R_0 \oplus R_1 = \langle \{x, y, z\}, D, \frac{x}{b} \mid \frac{y}{a} \mid \frac{z}{a} \rangle$.

x	y	z
a	b	a
a	b	b
b	a	a
b	a	b
a	a	a
b	b	a

Example 2

$$\pi_{\{y\}}(\langle \{x, y\}, \{a, b\}, \frac{x}{a} \mid \frac{y}{b} \rangle) = \langle \{y\}, \{a, b\}, \frac{y}{b} \rangle.$$

$$\rho_{\{x, y\}}(\langle \{y\}, \{a, b\}, \frac{y}{b} \rangle) = \langle \{x, y\}, \{a, b\}, \frac{x}{a} \mid \frac{y}{b} \rangle.$$

\bowtie is similar to the numerous variants of “join” in relational databases (natural join, equijoin, crossjoin, thajoin, and perhaps others). To prevent confusion, we refer to the \bowtie defined here as “bow tie”. \oplus is forced into existence as the counterpart of bow tie, and is new to us. For lack of a better term, we refer to it as “oplus”. “Projection” and “cylinder” are inspired by Tarski *et al.* [3].

Note that bow tie and oplus are generalizations of intersection and union. In the cylindric set algebra of [3] all relations have the same type (with the same infinite index set) so that the generalization represented by bow tie and oplus is not needed.

If the index set I is n for some ordinal n , then the relation is said to be an n -ary relation. It is a subset of D^n . It is said to be a type-P relation because it can be the interpretation of a predicate symbol.

If I is the set of free variables of an expression, then the relation is said to be type C. Thus the relation defined in Definition 7 on page 8 is a type-C relation. Here “C” is from “cylinder” because the cylinders in cylindric algebra [3] are relations of this kind. Type-C relations cannot be interpretations of predicate symbols, as the arguments in atomic formulas are indexed numerically.

Lemma 2 Let $R_0 = \langle I_0, D, C_0 \rangle$, $R_1 = \langle I_1, D, C_1 \rangle$, and $I = I_0 \cup I_1$. We have

$$\begin{aligned} R_0 \bowtie R_1 &= \rho_I(R_0) \cap \rho_I(R_1) \\ R_0 \oplus R_1 &= \rho_I(R_0) \cup \rho_I(R_1) \end{aligned}$$

Proof

Immediate from the definitions. □

Lemma 3

$$\begin{aligned}\langle I', D, C \rangle &= \pi_{I'}(\rho_I(\langle I', D, C \rangle)) \\ \langle I, D, C \rangle &\subseteq \rho_I(\pi_{I'}(\langle I, D, C \rangle))\end{aligned}$$

□

Proof

Note that the $\pi_{I'}$ and $\rho_I(\pi_{I'}$ of Definition 3 are respectively a set extension and inverse set extension (Definition 1) of function restriction (Definition 2). With this identification, the Lemma to be proved can be seen to follow from Lemma 1. □

We will encounter a situation involving a set X of n variables, a set D , and a tuple α of type $X \rightarrow D$. Let $x = (x_0, \dots, x_{n-1})$, a tuple of type $n \rightarrow X$, be an enumeration of X . That is, there is no repeated occurrence of any variable in $x = (x_0, \dots, x_{n-1})$. The tuple t defined as $x \triangleright \alpha$ is of type $n \rightarrow D$. We extend the function composition \triangleright from individual tuples to relations, that is, to sets of tuples of the same type.

Definition 4

$$\langle S, T, C_0 \rangle \triangleright \langle T, U, C_1 \rangle = \langle S, U, \{d_0 \triangleright d_1 \mid d_0 \in C_0 \wedge d_1 \in C_1\} \rangle.$$

With $d_0 \in \langle S \rightarrow T \rangle$ and $d_1 \in \langle T \rightarrow U \rangle$ we write $\langle S, T, C_0 \rangle \triangleright d_1$ for $\langle S, T, C_0 \rangle \triangleright \langle T, U, \{d_1\} \rangle$ and $d_0 \triangleright \langle T, U, C_1 \rangle$ for $\langle S, T, \{d_0\} \rangle \triangleright \langle T, U, C_1 \rangle$. □

Lemma 4 *Let S and T be sets, not both empty. We have*

$$\begin{aligned}S \rightarrow S &= (S \rightarrow T) \triangleright (T \rightarrow S) \\ S \rightarrow T &= (S \rightarrow S) \triangleright (S \rightarrow T) = (S \rightarrow T) \triangleright (T \rightarrow T)\end{aligned}$$

□

3 Satisfaction semantics

One aspect of semantics defines the conditions under which a sentence, i.e. a *closed* formula, is true in (is *satisfied* by) an interpretation. We first review this aspect of semantics, which we call *satisfaction semantics*. Then we show how it can also be applied to the definition of relations and functions.

3.1 Logic preliminaries

Definition 5 1. A signature consists of

- (a) A set of constant symbols.
- (b) Sets of n -ary predicate symbols for nonnegative integers n . We denote the arity of a predicate symbol q by $|q|$.
- (c) Sets of n -ary function symbols for positive integers n . We denote the arity of a function symbol f by $|f|$.

A term (formula) of first-order predicate logic is an L -term (L -formula) if the function and predicate symbols have the names and arities specified in L .

- 2. A structure consists of a universe D (also referred to as “domain”), which is a set, and numerically-indexed relations and functions over D .

For a given signature L , a structure S with universe D is an L -structure whenever

- (a) each constant in L is associated with an element of D ,
- (b) each predicate symbol p in L is associated with a relation in S of type $D^{|p|}$,
- (c) and each function symbol f in L is associated with a function in S of type $D^{|f|} \rightarrow D$.

3. Let a signature L and an L -structure S with domain D be given. An interpretation of an L -term T consists of a mapping from each function symbol f of T to an $|f|$ -ary function of S . An interpretation of an L -formula F consists of a mapping from each predicate symbol p of F to a $|p|$ -ary relation of S .

Note that $I(f)$ and $I(p)$ are numerically indexed.

3.2 Satisfaction

Given a signature L and an interpretation I for L -formulas. Let D be the universe of discourse of I . The truth value of an L -formula F with V as set of free variables depends on I for the interpretation of the predicate and function symbols. It also depends on an assignment α of individuals in D to the variables in V . That is, α is a function of type $V \rightarrow D$. With these dependencies in mind, we write $M_\alpha^I(F)$ for the meaning, that is a truth value, of formula F .

$M_\alpha^I(F)$ is an expression in the metalanguage, which is informal mathematics. Here F is the metalanguage name for a formula in first-order predicate logic.

The meaning of $M_\alpha^I(F)$, with F a formula, is a truth value. The meaning of $M_\alpha^I(t)$, with t a term, is an individual in the universe of discourse D . The meaning of $M_\alpha^I(t_0, \dots, t_{n-1})$ is an n -tuple of individuals in the universe of discourse D .

Several well-known textbooks [7, 10, 1, 2] define satisfaction semantics in substantial agreement; [7, 2] attribute the definition to Tarski [12], which has been translated in [14] and in [13]. Tarski's paper addresses philosophers and argues that the concept of truth can only be defined without danger of paradox in formalized languages and then only in a suitable class of these. By the 1960's Tarski's satisfaction semantics had become folklore to a sufficient extent that Shoenfield [10] and Enderton [1] give it without attribution.

Definition 6 Let L be a signature including at least one constant, a function symbol f , and a predicate symbol p . Let I be an interpretation for L with D as universe of discourse.

1. $M_\alpha^I(t) = I(t)$ if t is a constant.
2. $M_\alpha^I(t) = \alpha(t)$ if t is a variable.
3. $M_\alpha^I(t) = (I(f))(M_\alpha^I(t_0), \dots, M_\alpha^I(t_{n-1}))$ if t is $f(t_0, \dots, t_{n-1})$.
4. If p is a predicate symbol, then $M_\alpha^I(p(t_0, \dots, t_{n-1}))$ is true iff

$$(M_\alpha^I(t_0), \dots, M_\alpha^I(t_{n-1})) \in I(p).$$

5. $M_\alpha^I(\neg F)$ is true iff not $M_\alpha^I(F)$.
6. Let F_0 have V_0 as set of free variables, let F_1 have V_1 as set of free variables, and let $V = V_0 \cup V_1$. Let $\alpha \in V \rightarrow D$.
We define $M_\alpha^I(F_0 \wedge F_1)$ iff $M_{\alpha \downarrow V_0}^I(F_0)$ and $M_{\alpha \downarrow V_1}^I(F_1)$.
We define $M_\alpha^I(F_0 \vee F_1)$ iff $M_{\alpha \downarrow V_0}^I(F_0)$ or $M_{\alpha \downarrow V_1}^I(F_1)$.
7. Let F be a formula with set V of free variables. Let α be a tuple of type $V \rightarrow D$. Let, for some $\{x_0, \dots, x_{n-1}\} \subseteq V$ and some $d_i \in D$ for $i \in n$, $\alpha' \in (V \rightarrow D)$ be such that $\alpha'(x_i) = d_i$ for all $i \in n$ and $\alpha'(v) = \alpha(v)$ if $v \in V \setminus \{x_0, \dots, x_{n-1}\}$.

We define $M_{\alpha \downarrow (V \setminus \{x_0, \dots, x_{n-1}\})}^I (\exists x_0 \dots \exists x_{n-1}. F)$ iff there exist $d_0, \dots, d_{n-1} \in D$ such that $M_{\alpha'}^I(F)$.

We define $M_{\alpha \downarrow (V \setminus \{x_0, \dots, x_{n-1}\})}^I (\forall x_0 \dots \forall x_{n-1}. F)$ iff for all $d_0, \dots, d_{n-1} \in D$ it is the case that $M_{\alpha'}^I(F)$.

Conventionally, the language of first-order predicate logic has terms and formulas as its syntactic categories. To these we add n -tuples of terms. Their denotation is defined as follows.

$$M_{\alpha}^I(t_0, \dots, t_{n-1}) = (M_{\alpha}^I(t_0), \dots, M_{\alpha}^I(t_{n-1})) \in D^n.$$

□

We may write $\exists V'. F$ instead of $\exists x_0 \dots \exists x_{n-1}. F$ when $V' = \{x_0, \dots, x_{n-1}\}$ is a subset of the set of free variables of F . Similarly for universal quantification.

Lemma 5 *In the context of Definition 6 we have*

$$M_{\alpha}^I(f(t_0, \dots, t_{n-1})) = (I(f))M_{\alpha}^I(t_0, \dots, t_{n-1}).$$

□

Proof

Apply Definition 6.

□

3.3 Application of satisfaction semantics to the definition of relations

In his treatment of set theory, Suppes [11] gives the Axiom of Separation (Zermelo's "Aussonderungsaxiom") as

$$\exists y \forall x [x \in y \leftrightarrow (x \in z \wedge \varphi(x))] \quad (1)$$

where z is a given set. This can be paraphrased to

$$\text{if } z \text{ is a set, then } \{x \in z \mid \varphi(x)\} \text{ is a set.} \quad (2)$$

In this general formulation it is not specified in what language $\varphi(x)$ is expressed. We make $\varphi(x)$ more specific by expressing it as a formula F of first-order predicate logic. Let us call V the set of free variables in F . Then, given an interpretation I , the meaning of F is determined by an assignment α of elements of D to the variables in V . That is, in (2) the given set z becomes $V \rightarrow D$ and we get for the subset of z called into existence by the Axiom of Separation:

$$\{\alpha \in (V \rightarrow D) \mid F \text{ is true in } I \text{ with } \alpha\}, \quad (3)$$

which is, using the formalism developed in this section,

$$\{\alpha \in (V \rightarrow D) \mid M_{\alpha}^I(F)\} \quad (4)$$

We arrived at (4) in an attempt to clarify the Axiom of Separation by means of first-order predicate logic. This expression can also be used to define the meaning of a formula when an interpretation is given, abstracting away from the assignment α .

Definition 7 *The meaning of a formula F with set V of free variables, given interpretation I with domain D is written as $M^I(F)$ and defined to be $\{\alpha \in (V \rightarrow D) \mid M_\alpha^I(F)\}$.*

Definition 7 defines the meaning of F as a subset of $V \rightarrow D$. That is, as a set of tuples, which is the usual definition of a *relation*. A compositional semantics of first-order predicate logic investigates how the meanings of the subformulas of F are connected to the meaning of F itself. As these meanings are now seen to be relations, we need a suitable set of operations on relations. These were defined in Section 2.4.

3.4 Application of satisfaction semantics to the definition of functions

Tarski *et al.* [15, 3] defined certain relations as denotations of formulas with free variables. A natural counterpart would be to define functions as denotations of terms. However, this was omitted from [15, 3], as the language of logic there does not include function symbols. In this section we introduce a counterpart of Definition 7 that defines a function as denotation of any term. Such a definition is only interesting if the denotation of a composite term is related in a plausible way to the denotations of its constituent terms. This we do in Section 6.

Let t be a term with set V of variables. Let I be an interpretation for t with universe of discourse D . Every $\alpha \in (V \rightarrow D)$ determines $M_\alpha^I(t) \in D$. In other words, I and t determine a function of type $(V \rightarrow D) \rightarrow D$. Hence we define the meaning $M^I(t) \in ((V \rightarrow D) \rightarrow D)$ of t as a function in the following way.

Definition 8 *Given an interpretation I with domain D . Let t be a term with set V of variables. We write the meaning of t as $M^I(t)$ and define it as the function in $(V \rightarrow D) \rightarrow D$ with map $\alpha \mapsto M_\alpha^I(t)$.* \square

4 Compositional semantics of composite formulas

Lemma 6 $M_\alpha^I(F)$ iff $\alpha \in M^I(F)$ iff $\alpha \in \{\beta \in (V \rightarrow D) \mid M_\beta^I(F)\}$.

Definition 9 *Let t_0, \dots, t_{n-1} be a tuple of terms with set V of variables. Let I be an interpretation for t_0, \dots, t_{n-1} with universe D . We define*

$$M^I(t_0, \dots, t_{n-1}) = \{M_\alpha^I(t_0, \dots, t_{n-1}) \mid \alpha \in (V \rightarrow D)\}.$$

\square

We provide examples and lemmas to build intuition about the consequences of Definitions 7 and 9.

Example 3 *In Definition 9 one may regard the tuples of terms as a language to specify subsets of $n \rightarrow D$, alias relations of type $n \rightarrow D$. For example $M^I(x, \dots, x)$ should denote the diagonal of the set diagonals in $n \rightarrow D$.*

$$\begin{aligned} M^I(x, \dots, x) &= (1) \\ \{M_\alpha^I(x, \dots, x) \mid \alpha \in (\{x\} \rightarrow D)\} &= (2) \\ \{(M_\alpha^I(x), \dots, M_\alpha^I(x)) \mid \alpha \in (\{x\} \rightarrow D)\} &= (3) \\ \{(\alpha(x), \dots, \alpha(x)) \mid \alpha \in (\{x\} \rightarrow D)\} &= (4) \\ \{(d, \dots, d) \mid d \in D\}. \end{aligned}$$

(1) Definition 9; (2) Lemma 5; (3) Definition 6; (4) let $d = \alpha(x)$ and note that for every $d \in D$ there is exactly one $\alpha \in (\{x\} \rightarrow D)$ such that $\alpha(x) = d$. \square

Example 4 Let I be an interpretation with universe of discourse D . Let $\{x_0, \dots, x_{n-1}\}$ be an enumeration of a set V of variables. In $M^I(x_0, \dots, x_{n-1}) \subseteq (n \rightarrow D)$ the arguments are at their least constrained, so this expression should denote all of $n \rightarrow D$.

$$\begin{aligned}
M^I(x_0, \dots, x_{n-1}) &= (1) \\
\{(M_\alpha^I(x_0), \dots, M_\alpha^I(x_{n-1})) \mid \alpha \in (V \rightarrow D)\} &= (2) \\
\{(\alpha(x_0), \dots, \alpha(x_{n-1})) \mid \alpha \in (V \rightarrow D)\} &= (3) \\
\{(\alpha(x_0), \dots, \alpha(x_{n-1})) \mid (x_0, \dots, x_{n-1}) \triangleright \alpha \in ((x_0, \dots, x_{n-1}) \triangleright (V \rightarrow D))\} &= (4) \\
\{(d_0, \dots, d_{n-1}) \mid (x_0, \dots, x_{n-1}) \triangleright \alpha \in ((x_0, \dots, x_{n-1}) \triangleright (V \rightarrow D))\} &= (5) \\
\{(d_0, \dots, d_{n-1}) \mid (x_0, \dots, x_{n-1}) \triangleright \alpha \in (n \rightarrow D)\} &= (6) \\
\{(d_0, \dots, d_{n-1}) \mid (d_0, \dots, d_{n-1}) \in (n \rightarrow D)\} &= (7) \\
n \rightarrow D.
\end{aligned}$$

(1) Definition 9 and Lemma 5; (2) Definition 6; (3) rewrite condition; (4) introduce the names d_i for $\alpha(x_i)$; (5) Definition 4; (6) definition of function composition; (7) meaning of set comprehension. \square

The compositional semantics of conjunction and disjunction is given by Theorem 1.

Theorem 1 Let F_0 and F_1 be formulas. Let I be an interpretation for the predicate and function symbols of F_0 and F_1 . We have

- (a) $M^I(F_0 \wedge F_1) = M^I(F_0) \bowtie M^I(F_1)$
- (b) $M^I(F_0 \vee F_1) = M^I(F_0) \oplus M^I(F_1)$ and
- (c) $M^I(\neg F) = M^I(F)^\sim$.

Proof

Let V_0 and V_1 be the sets of free variables of F_0 and F_1 , respectively. Let D be the domain of I .

$$\begin{aligned}
M^I(F_0 \wedge F_1) &= (1) \\
\{\alpha \in ((V_0 \cup V_1) \rightarrow D) \mid M_\alpha^I(F_0 \wedge F_1)\} &= (2) \\
\{\alpha \in ((V_0 \cup V_1) \rightarrow D) \mid M_{\alpha \downarrow V_0}^I(F_0) \text{ and } M_{\alpha \downarrow V_1}^I(F_1)\} &= (3) \\
\{\alpha \in ((V_0 \cup V_1) \rightarrow D) \mid \alpha \downarrow V_0 \in \{\beta \in V_0 \mid M_\beta^I(F_0)\} \text{ and } \alpha \downarrow V_1 \in \{\gamma \in V_1 \mid M_\gamma^I(F_1)\}\} &= (4) \\
\{\alpha \in ((V_0 \cup V_1) \rightarrow D) \mid \alpha \downarrow V_0 \in M^I(F_0) \text{ and } \alpha \downarrow V_1 \in M^I(F_1)\} &= (5) \\
M^I(F_0) \bowtie M^I(F_1).
\end{aligned}$$

(1): Definition 7, (2): Definition 6, (3): Lemma 6, (4): meaning of set comprehension, and (5): Definition 3.

There is a similar proof of part (b). Proof of (c):

$$\begin{aligned}
M^I(\neg F) &= (1) \\
\{\alpha \in (V \rightarrow D) \mid M_\alpha^I(\neg F)\} &= (2) \\
\{\alpha \in (V \rightarrow D) \mid \text{not } M_\alpha^I(F)\} &= (3) \\
(V \rightarrow D) \setminus \{\alpha \in (V \rightarrow D) \mid M_\alpha^I(F)\} &= (4) \\
(V \rightarrow D) \setminus M^I(F).
\end{aligned}$$

(1) Definition 7, (2) Definition 6, (3) meaning of set comprehension, (4) Definition 7. \square

This is basically a result from [3], where conjunction corresponds to intersection of relations. The difference arises because there all relations have the same index set; in that special case bow tie and intersection coincide, as do plus and union.

Theorem 2 Let F be a formula with V as set of free variables. Let I be an interpretation for the predicate and function symbols with D as universe of discourse. Let V' be a subset of V . We have $M^I(\exists V'. F) = \pi_{V \setminus V'} M^I(F)$.

Proof

$$\begin{aligned}
M^I(\exists V'. F) &= (1) \\
\{\alpha' \in ((V \setminus V') \rightarrow D) \mid M_{\alpha'}^I(\exists V'. F)\} &= (2) \\
\{\alpha' \in ((V \setminus V') \rightarrow D) \mid \exists \alpha \in (V \rightarrow D) \text{ s.t. } M_{\alpha}^I(F) \text{ and } \alpha \downarrow (V \setminus V') = \alpha'\} &= (3) \\
\{\alpha \downarrow (V \setminus V') \mid \alpha \in M^I(F)\} &
\end{aligned}$$

(1) Definition 7, (2) Definition 6, and (3) Definition 7.

This completes the proof, as the final expression is the content of the relation $\pi_{V \setminus V'} M^I(F)$ according to Definition 3. \square

5 Compositional semantics of atomic formulas

Tarski *et al.* [15, 3] introduced denotations for logic formulas only as motivation for the introduction of cylindric algebras. They did not consider decomposition of atomic formulas. Our primary interest is compositional semantics. This point of view demands a semantic analysis of the inner structure of atomic formulas.

$M^I(p(t_0, \dots, t_{n-1}))$ is determined by Definition 7. Yet it is also determined by the denotation of its constituent p (according to I) and by that of its constituent (t_0, \dots, t_{n-1}) (according to Definition 9). The following theorem shows how $M^I(p(t_0, \dots, t_{n-1}))$ is determined by the denotations of its constituents.

Theorem 3 *Let L be a signature with a predicate symbol p . Let I be an L -interpretation with universe of discourse D . Let t_0, \dots, t_{n-1} be L -terms with V as union of the sets of their variables.*

We have

$$M^I(p(t_0, \dots, t_{n-1})) = (V \rightarrow n) \triangleright (I(p) \cap M^I(t_0, \dots, t_{n-1})).$$

Proof (1) Lemma 4,

$$\begin{aligned}
(V \rightarrow n) \triangleright (I(p) \cap M^I(t_0, \dots, t_{n-1})) &= (1) \\
(V \rightarrow n) \triangleright (I(p) \cap \{M_{\alpha}^I(t_0, \dots, t_{n-1}) \mid \alpha \in (V \rightarrow D)\}) &= (2) \\
(V \rightarrow n) \triangleright (I(p) \cap \{d \in (n \rightarrow D) \mid \exists \alpha \in (V \rightarrow D). d = M_{\alpha}^I(t_0, \dots, t_{n-1})\}) &= (3) \\
(V \rightarrow n) \triangleright \{d \in (n \rightarrow D) \mid \exists \alpha \in (V \rightarrow D). d \in I(p) \wedge d = M_{\alpha}^I(t_0, \dots, t_{n-1})\} &= (4) \\
(V \rightarrow n) \triangleright \{d \in (n \rightarrow D) \mid \exists \alpha \in (V \rightarrow D). M_{\alpha}^I(p(t_0, \dots, t_{n-1}))\} &= (5) \\
\{\gamma \triangleright d \mid \gamma \in (V \rightarrow n) \wedge d \in (n \rightarrow D) \wedge \exists \alpha \in (V \rightarrow D) \wedge M_{\alpha}^I(p(t_0, \dots, t_{n-1}))\} &= (6) \\
\{\alpha \in (V \rightarrow D) \mid M_{\alpha}^I(p(t_0, \dots, t_{n-1}))\} &= (7) \\
M^I(p(t_0, \dots, t_{n-1})). &
\end{aligned}$$

(1) Definition 9

(2) note that the set comprehension expression is the range of the function

$\lambda \alpha \in (V \rightarrow D). M_{\alpha}^I(t_0, \dots, t_{n-1})$ and use the fact that

$\{f(x) \mid x \in S\} = \{y \in T \mid \exists x \in S. y = f(x)\}$

(3) eliminate the set intersection with $I(p)$ by inserting the equivalent condition $d \in I(p)$

(4) Definition 6

(5) Definition 4

(6) give $\gamma \triangleright d$ the name α' ; this eliminates γ and d , and identify α' with the α that exists according to $\exists \alpha \in (V \rightarrow D)$ in the condition

(7) Definition 7. \square

Theorem 4 *Let L be a signature containing a predicate symbol p . Let I be an L -interpretation with universe of discourse D . Let t_0, \dots, t_{n-1} be L -terms with V as union of their sets of variables.*

We have

$$(n \rightarrow V) \triangleright M^I(p(t_0, \dots, t_{n-1})) = I(p) \cap M^I(t_0, \dots, t_{n-1}).$$

Proof

$$\begin{aligned} I(p) \cap M^I(t_0, \dots, t_{n-1}) &= (1) \\ (n \rightarrow n) \triangleright (I(p) \cap M^I(t_0, \dots, t_{n-1})) &= (2) \\ ((n \rightarrow V) \triangleright (V \rightarrow n)) \triangleright (I(p) \cap M^I(t_0, \dots, t_{n-1})) &= (3) \\ (n \rightarrow V) \triangleright ((V \rightarrow n)) \triangleright (I(p) \cap M^I(t_0, \dots, t_{n-1})) &= (4) \\ (n \rightarrow V) \triangleright M^I(p(t_0, \dots, t_{n-1})). & \end{aligned}$$

(1): Lemma 4, (2): Lemma 4, (3): Lemma 4, (4): Theorem 3. \square

Corollary 1 *Let L be a signature including predicate symbol p . Let I be an L -interpretation with D as universe of discourse. Let $\{x_0, \dots, x_{n-1}\}$ be a set of n variables. We have*

$$(n \rightarrow \{x_0, \dots, x_{n-1}\}) \triangleright M^I(p(x_0, \dots, x_{n-1})) = I(p).$$

Proof

$$\begin{aligned} (n \rightarrow \{x_0, \dots, x_{n-1}\}) \triangleright M^I(p(x_0, \dots, x_{n-1})) &= (1) \\ I(p) \cap M^I((x_0, \dots, x_{n-1})) &= (2) \\ I(p) \cap (n \rightarrow D) &= (3) \\ I(p). & \end{aligned}$$

(1) Theorem 4, (2) Example 4, and (3) $I(p) \subseteq (n \rightarrow D)$. \square

Informally it is clear that the role of the arguments in $p(t_0, \dots, t_{n-1})$ is to restrict the extent of the relation denoted by p . In Corollary 1 we see that if (t_0, \dots, t_{n-1}) is as unrestrictive as possible, then we get all of $I(p)$ back. In Example 3 we see that if (t_0, \dots, t_{n-1}) is as restrictive as possible, then we get back only a diagonal slice of p .

In Theorem 4 the right-hand side is an expression of the intuition of the arguments restricting the meaning of the relation denoted by the predicate symbol. Naively, the right-hand side should equal $M^I(p(t_0, \dots, t_{n-1}))$, but this is a C -type relation, whereas the right-hand side is a P -type relation. The simplest fix of the discrepancy that is at least *prima facie* correct is the composition in the left-hand side. That a simple fix yields a correct formula is encouraging.

6 Compositional semantics of terms

$M^I(f(t_0, \dots, t_{n-1}))$ is determined by Definition 8. Yet it is also determined by the denotation of its constituent f (according to I) and by that of its constituent (t_0, \dots, t_{n-1}) (according to Definition 9). The following shows the way in which $f(t_0, \dots, t_{n-1})$ is determined by the denotations of its constituents.

Theorem 5 *Let L be a signature with a function symbol f . Let I be an L -interpretation with universe of discourse D . Let t_0, \dots, t_{n-1} be L -terms with V as union of the sets of their variables.*

We assert that for all $\alpha \in (V \rightarrow D)$ there exists an $x \in (n \rightarrow V)$ such that

$$(M^I(f(t_0, \dots, t_{n-1}))) (\alpha) = (I(f) \downarrow M^I(t_0, \dots, t_{n-1})) (x \triangleright \alpha).$$

See Figure 1.

Proof.

With $b = (M^I(f(t_0, \dots, t_{n-1}))) (\alpha)$ we have

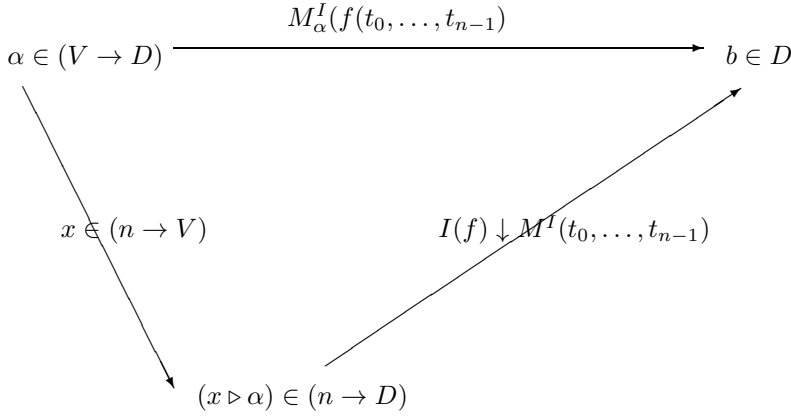


Figure 1: Diagram to illustrate Theorem 5, which asserts that the triangle commutes.

$$M^I(f(t_0, \dots, t_{n-1}))(\alpha) = (1)$$

$$(\lambda \beta \in (V \rightarrow D). M_\beta^I(f(t_0, \dots, t_{n-1}))) (\alpha) = (2)$$

$$M_\alpha^I(f(t_0, \dots, t_{n-1})) = (3)$$

$$(I(f))(M_\alpha^I(t_0), \dots, M_\alpha^I(t_{n-1})) = b.$$

(1): Definition 7, (2): beta reduction, and (3): Definition 6.

Furthermore,

$$(I(f))(M_\alpha^I(t_0), \dots, M_\alpha^I(t_{n-1})) = b \quad \text{iff} \quad (1)$$

$$\exists d \in (n \rightarrow D). (I(f))(d) = b \wedge d = (M_\alpha^I(t_0), \dots, M_\alpha^I(t_{n-1})) \wedge d = x \triangleright \alpha \quad \text{iff} \quad (2)$$

$$\exists d \in (n \rightarrow D). (I(f))(d) = b \wedge d \in M^I(t_0, \dots, t_{n-1}) \wedge d = x \triangleright \alpha \quad \text{iff} \quad (3)$$

$$(I(f) \downarrow M^I(t_0, \dots, t_{n-1}))(x \triangleright \alpha) = b.$$

(1): $I(f)$ is a function in $(n \rightarrow D) \rightarrow D$, (2): Definition 9, and (3): Definition 2 for function restriction. \square

7 Conclusions

In 1933 Tarski identified first-order predicate logic [12] as a language in which the concept of truth can be defined mathematically. The definition Tarski gave there, though mathematical, still fell short of the standard set by the semantics of propositional logic. This semantics can be specified as a homeomorphism from the syntax algebra of propositions to the semantic algebra of truth values. This homeomorphism is not an isolated example, witness the following quote from [6]:

A technical description of the standard interpretation [of compositionality] is that syntax and semantics are algebras, and meaning assignment is a homomorphism from syntax to semantics. This definition of compositionality is found with authors such as Montague [8], Janssen [5], and Hodges [4].

Compositional semantics for first-order predicate logic is described in [15, 3]. This semantic algebra, the cylindric set algebra, has relations as carrier. These relations can be regarded as generalized truth values.

To us the most attractive feature of the algebraic approach is its compositional nature: the meaning of a composite expression is the result of a set-theoretically defined operation on the meanings of its components. In the existing work the decompositions thus treated were conjunction, disjunction, negation and quantification. There is the additional limitation that function symbols are absent. In this paper we give compositional semantics for the full language of first-order predicate logic, including function symbols. Moreover, we decompose atomic formulas and terms into their constituent predicate or function symbols and tuples of arguments.

The original Tarskian semantics specifies the conditions under which a sentence is satisfied by an interpretation. An interpretation assigns a relation as meaning to each predicate symbol, a function as meaning to each function symbol. In the algebraic semantics of Tarski et. al. there is no role for interpretation, so it is not possible to decompose the meaning of an atomic formula: there is no meaning assigned to the bare predicate symbol. In this paper we take interpretations into account and we give set-theoretic counterparts for conjunction, disjunction, negation, quantification, as well as application of predicate and function symbols to their arguments.

It is not surprising that our semantic structure is not the same as the one in the algebraic semantics of Tarski et. al. Both structures can be described in terms of two parameters: a set I of indexes and a set D which is the algebraic counterpart of the universe of discourse. In cylindric algebra semantics the carrier consists of the subsets of $I \rightarrow D$. In our case it is the set of all relations of type $I' \rightarrow D$ with $I' \subseteq I$. In Definition 3 we identified the most important operations. In [9] many algebras for quantifier logics are described. For the purpose of this paper it does not matter which, if any, of these matches our carrier and set of operations.

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References

- [1] H.B. Enderton. *A Mathematical Introduction to Logic*. Academic Press, 1972.
- [2] A. Grzegorzczk. *An Outline of Mathematical Logic*. D. Reidel Publishing Company, 1974.
- [3] Leon Henkin, J Donald Monk, and Alfred Tarski. *Cylindric Algebras, Part I*. North-Holland, 1971.
- [4] Wilfrid Hodges. Formal features of compositionality. *Journal of Logic, Language and Information*, 10(1):7–28, 2001.
- [5] Theo M.V. Janssen. Compositionality. In Johan van Benthem and Alice ter Meulen, editors, *Handbook of Logic and Language*, pages 417–473. MIT Press, 1997.
- [6] Theo M.V. Janssen. Frege, contextuality and compositionality. *Journal of Logic, Language and Information*, 10(1):115–136, 2001.
- [7] E. Mendelson. *An Introduction to Mathematical Logic*. Van Nostrand, 1964.
- [8] R. Montague. Universal grammars. *Theoria*, 36:373–398, 1970. Reprinted in [16].

- [9] István Németi. Algebraization of quantifier logics, an introductory overview. *Studia Logica*, 50(3-4):485–569, 1991.
- [10] Joseph R. Shoenfield. *Mathematical Logic*. Addison-Wesley, 1967.
- [11] Patrick Suppes. *Axiomatic Set Theory*. Van Nostrand, 1960.
- [12] A. Tarski. The notion of truth in the languages of the deductive sciences (in Polish). C.R. Soc. Sci, Varsovie, Cl. III, 1933.
- [13] A. Tarski. *Logic, Semantics, Metamathematics*. Oxford University Press, 1956.
- [14] Alfred Tarski. Der Wahrheitsbegriff in den formalisierten Sprachen. *Studia Philosophica*, 1, 1935.
- [15] Alfred Tarski. A representation theorem for cylindric algebras. In *Bulletin of the American Mathematical Society*, volume 58, pages 65–66, 1952.
- [16] Richard H. Thomason. *Formal Philosophy: Selected Papers of Richard Montague*. Yale University Press, 1974.
- [17] M.H. van Emden. Compositional semantics for the procedural interpretation of logic. In S. Etalle and M. Truszczyński, editors, *Proc. Intern. Conf. on Logic Programming*, number 4079 in Springer Lecture Notes in Computer Science, pages 315 – 329. Springer Verlag, 2006.